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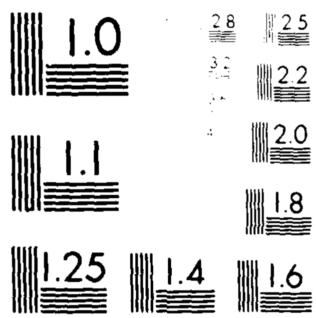

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SENSITIVITY AND ASYMPTOTIC PROPERTIES OF BAYESIAN RELIABILITY ESTIMATES

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ABSTRACT

In decision theoretic approaches to estimation problems, loss functions of the type $L(\theta,\hat{\theta}) = |\theta-\hat{\theta}|^{\alpha}$, $\alpha > 0$ are often employed; often $\alpha = 2$. In many applications of reliability and life testing, such loss functions are inappropriate. Alternative loss functions which appear to be more suited to the intended application are proposed and Bayesian estimates of the exponential parameter are obtained for these.

Asymptotic expansions of such estimators are given and compared with estimators given in the literature.

AMS (MOS) Subject Classification: 62N05

Key Words: Reliability, Life testing, Bayesian estimation, Large sample theory.

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SIGNIFICANCE AND EXPLANATION

This report studies the sensitivity of reliability estimates to changes in the loss function. The choice of loss function is an important consideration in assessing the merits of an estimator, since this embodies the concerns of the applier in terms of those errors of estimation for which he would like to insure the greatest protection.

The sensitivity is examined by means of asymptotic analyses which permit one to ascertain readily the deviations from the maximum likelihood estimator.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SENSITIVITY AND ASYMPTOTIC PROPERTIES OF

BAYESIAN RELIABILITY ESTIMATES

Bernard Harris
&
Andrew P. Soms

1. INTRODUCTION

In decision theoretic approaches to estimation problems, the error of estimation is typically measured by loss functions of the form

$$L(\theta,\hat{\theta}) = |\theta-\hat{\theta}|^{\nu}, \quad \nu > 0 ,$$

where θ is the unknown parameter and $\hat{\theta}$ is the corresponding estimator. However, as the following heuristic argument will demonstrate, such a loss function is oten inappropriate for many applications of reliability and life testing.

Let R, $0 \le R < 1$, be the reliability of some device. If $\hat{R}$ is an estimator of R and the true value of R is .5, then an error of the magnitude $|\hat{R}-R| = .1$ would be unlikely to affect any administrative decision concerning the feasibility of the device, since a device whose reliability is between .4 and .6 would not usually be regarded as satisfactory. On the other hand, if $R = .90$, then one device in ten fails and the estimate $\hat{R} = .99$ would suggest that only one device in 100 would fail. Thus, one might be inclined to conclude that one-tenth as many replacements were needed as were in fact required and consequently seriously misjudge maintenance and replacement costs. A similar but opposite error in judgement occurs when $R = .99$ and $\hat{R} = .90$. Consequently, it appears to be desirable to concentrate on errors of estimation for R close to unity.

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Considerations such as those described above suggest that loss functions suitable for reliability applications might have forms such as

$$L(R,\hat{R}) = |R-\hat{R}|^{\nu}/(1-R)^{\kappa}, \quad 0 \le R < 1,\ 0 \le \hat{R} < 1,\quad \nu > 0,\ \kappa > 0,$$

or

$$L(R,\hat{R}) = \left|\frac{1}{1-R} - \frac{1}{1-\hat{R}}\right|^{\nu}, \quad 0 \le R < 1, \quad 0 \le \hat{R} < 1, \quad \nu > 0.$$

In order to study the effects of using such loss functions, we consider a Bayesian model for exponential life testing and examine the asymptotic behavior of the estimates thus obtained.

Specifically let $X_1, X_2, \ldots, X_N$ be independent identically distributed observations from an exponential distribution with probability density function

$$f(x;\theta) = \theta e^{-\theta x}, \quad x > 0, \quad \theta > 0. \tag{1}$$

Then the reliability R_T is defined as

$$R_T = e^{-\theta T}, \tag{2}$$

where $T > 0$ is a constant assigned in advance of the experiment and known as the <u>mission time</u>. With no loss of generality, we can set $T = 1$, since this corresponds to selecting the scale with respect to which the observations $X_1, X_2, \ldots, X_N$ are measured.

In Section 2, we obtain Bayesian estimators for specific loss functions of the type described previously, when R has a beta distribution prior. We compare this with the Bayesian estimator of R that is obtained by assigning a gamma prior to θ, a choice of Bayesian model quite often found in the literature. Obviously, since R is a function of θ, this induces a prior

on R; however, the priors assigned to θ in such studies do not induce the beta family of priors on R.

In Section 3, the asymptotic behavior of these estimators is given. These asymptotic expressions facilitate studying the sensitivity of the estimators to the changes in the loss functions.

Section 4 is devoted to comparing the results obtained herein with previous estimation techniques.

Several appendices which provide the technical details needed to establish the existence of the estimators and the calculation of the asymptotic expansions are included at the end of the paper.

2. BAYESIAN ESTIMATION FOR UNCENSORED AND TYPE II CENSORED DATA

Let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(N)}$ be the order statistics from a random sample disbributed by (1). Assume that only the first n order statistics have been observed, $1 \le n \le N$. Then, it is well-known that the total time on test statistic Y, defined by

$$Y = \sum_{i=1}^{n} X_{(i)} + (N-n)X_{(n)} \tag{3}$$

is a sufficient statistic and its probability density function is

$$f_Y(y;\theta) = \theta^n e^{-\theta y} y^{n-1}/\Gamma(n), \quad y > 0,\ \theta > 0. \tag{4}$$

To represent (4) in terms of the reliability R, we reparametrize, obtaining

$$f_Y(y;R) = (-\log R)^n R^y y^{n-1}/\Gamma(n), \quad 0 < R < 1, \quad y > 0. \tag{5}$$

Let

$$\tau(R;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} R^{\alpha-1} (1-R)^{\beta-1}, \quad 0 < R < 1,\ \alpha > 0,\ \beta > 0, \tag{6}$$

be the prior distribution on R. Then the joint distribution of R and Y is given by

$$f(y,R;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n)} R^{\alpha+y-1}(1-R)^{\beta-1}(-\log R)^n y^{n-1}, \tag{7}$$

$0 < R < 1$, $0 < y$, $\alpha > 0$, $\beta > 0$, $1 \le n \le N$.

We now obtain the marginal distribution of Y. Expanding $(1-R)^{\beta-1}$ in a binomial series, we obtain

$$f_1(y;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n)} y^{n-1} \int_0^1 R^{\alpha+y-1}(-\log R)^n \sum_{i=0}^{\infty} \binom{\beta-1}{i}(-1)^i R^i dR. \tag{8}$$

In (8), it is understood that if β is a positive integer, the series

terminates; that is, all terms with $i > \beta - 1$ are zero.

The interchange of summation and integration in (8) is justifiable. Hence, we can write (8) as

$$f_1(y;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n)}\, y^{n-1} \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} \int_0^1 R^{\alpha+y+i-1}(-\log R)^n\, dR$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n)}\, y^{n-1} \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} \int_0^\infty \theta^n e^{-\theta(\alpha+y+i)} d\theta$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n)}\, y^{n-1} \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} \frac{\Gamma(n+1)}{(\alpha+y+i)^{n+1}}$$

$$= \frac{n\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, y^{n-1} \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} / (\alpha+y+i)^{n+1}. \tag{9}$$

In particular, the integration given in (9) is valid over a larger range of the parameters than specified in (7). Specifically

$$\int_0^1 R^{\alpha+y-1}\,(1-R)^{\beta-1}(-\log R)^n\, dR = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} \frac{\Gamma(n+1)}{(\alpha+y+i)^{n+1}}, \tag{10}$$

whenever $\alpha + y > 0$, $\beta + n > 0$. These facts will be subsequently utilized. The reader is referred to Appendix I for various details relevant to these remarks and calculations.

Some particular cases of (10) are worth noting. If $\beta = 0$, we have

$$\int_0^1 R^{\alpha+y-1}\,(1-R)^{-1}(-\log R)^n\, dR = \Gamma(n+1)\zeta(n+1,\alpha+y) \tag{11}$$

where $\zeta(r,s)$ denotes the generalized Riemann zeta function and $\zeta(r,1) = \zeta(r)$ is the Riemann zeta function.

Combining the results of (7) and (9), we can write down the posterior distribution of R given $Y = y$ as

$$f(R|Y=y) = \frac{R^{\alpha+y-1}\ (1-R)^{\beta-1}(-\log\ R)^n}{\Gamma(n+1)\ \sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}/(\alpha+y+i)^{n+1}} \quad . \tag{12}$$

This enables us to calculate the Bayes estimators for a number of the loss functions of the type described previously.

For $L_1(R,\hat{R}) = (R-\hat{R})^2$, the Bayes estimator

$$\hat{R}_{\alpha,\beta}(L_1) = E(R|Y=y) = \frac{\int_0^1 R^{\alpha+y}\ (1-R)^{\beta-1}(-\log\ R)^n\ dR}{\int_0^1 R^{\alpha+y-1}(1-R)^{\beta-1}(-\log\ R)^n\ dR} \tag{13}$$

or equivalently,

$$\hat{R}_{\alpha,\beta}(L_1) = \frac{\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}/(\alpha+y+i+1)^{n+1}}{\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}/(\alpha+y+i)^{n+1}} \quad . \tag{14}$$

In particular, for $\beta = 1$

$$\hat{R}_{\alpha,\beta}(L_1) = \left(1 - \frac{1}{\alpha+y+1}\right)^{n+1} \quad . \tag{15}$$

Similarly, for $L_{2,\nu}(R,\hat{R}) = (R-\hat{R})^2/(1-R)^{\nu}$, $\nu > 0$, the Bayes estimator is

$$\hat{R}_{\alpha,\beta,\nu}(L_2) = \frac{E\{R/(1-R)^{\nu}|Y=y\}}{E\{1/(1-R)^{\nu}|Y=y\}} \quad , \tag{16}$$

or

$$\hat{R}_{\alpha,\beta,\nu}(L_2) \quad \frac{\sum_{i=0}^{\infty} (-1)^i \binom{\beta-\nu-1}{i} /(\alpha+y+i+1)^{n+1}}{\sum_{i=0}^{\infty} (-1)^i \binom{\beta-\nu-1}{i} /(\alpha+y+i)^{n+1}} \;, \quad \beta-\nu+n>0 \;. \tag{17}$$

Finally, we consider the loss function $L_3(R,\hat{R}) = ((1-R)^{-1}-(1-\hat{R})^{-1})^2$. Here the Bayes estimator is given by

$$\hat{R}_{\alpha,\beta}(L_3) = 1 - (E\{(1-R)^{-1}|Y=y\})^{-1} \;. \tag{18}$$

Since

$$E\{(1-R)^{-1}|Y-y\} = \frac{\int_0^1 R^{\alpha+y-1}(1-R)^{\beta-2} \, (-\log R)^n \, dR}{\Gamma(n+1) \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i}/(\alpha+y+i)^{n+1}} \;, \tag{19}$$

and since the integral (19) converges whenever $\beta + n - 1 > 0$, we have

$$E\{(1-R)^{-1}|Y-y\} = \frac{\sum_{i=0}^{\infty} (-1)^i \binom{\beta-2}{i}/(\alpha+y+i)^{n+1}}{\sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i}/(\alpha+y+i)^{n+1}} \;. \tag{20}$$

Then, we can write

$$\hat{R}_{\alpha,\beta}(L_3) = \frac{\sum_{i=0}^{\infty} (-1)^i \binom{\beta-2}{i}/(\alpha+y+i)^{n+1} - \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i}/(\alpha+y+i)^{n+1}}{\sum_{i=0}^{\infty} (-1)^i \binom{\beta-2}{i}/(\alpha+y+i)^{n+1}} \;. \tag{21}$$

Applying the Pascal triangle identity, we have

$$\hat{R}_{\alpha,\beta}(L_3) = \frac{\sum_{i=0}^{\infty} (-1)^{i-1}\binom{\beta-2}{i-1}/(\alpha+y+i)^{n+1}}{\sum_{i=0}^{\infty} (-1)^{i}\binom{\beta-2}{i}/(\alpha+y+i)^{n+1}}$$

$$= \frac{\sum_{i=0}^{\infty} (-1)^{i}\binom{\beta-2}{i}/(\alpha+y+i+1)^{n+1}}{\sum_{i=0}^{\infty} (-1)^{i}\binom{\beta-2}{i}/(\alpha+y+i)^{n+1}} \quad . \tag{22}$$

Comparing (14), (17) and (22), we have

$$\hat{R}_{\alpha,\beta,\nu}(L_2) = \hat{R}_{\alpha,\beta-\nu}(L_1) \tag{23}$$

and

$$\hat{R}_{\alpha,\beta}(L_3) = \hat{R}_{\alpha,\beta-1}(L_1) \ . \tag{24}$$

Thus, we have shown that the three families of estimators have the same form. Hence, it is possible to study all of them simultaneously and a single asymptotic expansion, given in the appendix, permits simultaneous analysis of their asymptotic properties.

One caution should be noted. We have listed the posterior means whenever the posterior means exist. However, in some of these instances the Bayes risk will be infinite. Specifically, for the loss function L_2, the Bayes risk exists whenever $\beta + n - 2 > 0$.

In the statistical literature, one frequently finds the following Bayes estimator of R (see for example, N. R. Mann, R. E. Schafer, N. D. Singpurwalla [6], p. 398). This is the estimator obtained by using the loss function

$L_1(R,\hat{R})$ and assigning a gamma distribution prior to θ, the parameter in the exponential density (1). A brief sketch of the calculation of the estimator follows. The prior is given by

$$\tau(\theta:\gamma,\delta) = \delta^{\gamma}\, \theta^{\gamma-1}\, e^{-\delta\theta}/\Gamma(\gamma)\ , \qquad (25)$$

where $\delta,\gamma > 0$ and $\theta > 0$. Then, calculating the posterior distribution of θ, when the data is given by the total time on test statistic (3), it follows that the estimator

$$\hat{R}(\gamma,\delta) = \left(\frac{y+\delta}{y+\delta+1}\right)^{n+\gamma}\ , \qquad (26)$$

the conditional expected value of R given $Y = y$. Subsequently, this will be compared with (14), (17) and (22).

Remark. To the authors, one of the more interesting properties of these estimators is the role played by the Riemann zeta function and the generalized Riemann zeta function. From (11), we have that

$$\hat{R}_{\alpha,0}(L_1) = \zeta(n+1,\ \alpha+y+1)\ /\ \zeta(n+1,\alpha+y)$$

$$= \hat{R}_{\alpha,\beta,\beta}(L_2) = \hat{R}_{\alpha,1}(L_3)\ .$$

Further, from (21), we see that

$$\hat{R}_{\alpha,1}(L_3) = (\zeta(n+1,\alpha+y) - (\alpha+y)^{-n-1})\ /\ \zeta(n+1,\alpha+y)$$

and if $\alpha = 1$, $y = 0$,

$$\hat{R}_{1,1}(L_3) = (\zeta(n+1) - 1)\ /\ \zeta(n+1)\ .$$

Specifically, if in addition $n = 1$,

$$\hat{R}_{1,1}(L_3) = (\pi^2/6 - 1)/(\pi^2/6) = 1 - \frac{6}{\pi^2} .$$

We conclude this section with some observations concerning elementary properties of the estimators given by (14), (17) and (22). With no loss of generality, we can examine specifically (13) and (14). From (13), it is immediately evident that $0 < \hat{R} < 1$. We now show that $\hat{R}$ is an increasing function of α (or y) and a decreasing function of β, which one would naturally expect to be the case, given the respective roles played by α and β in the model. Therefore, we calculate

$$\frac{\partial \hat{R}}{\partial \beta} = \frac{\int_0^1 R^{\alpha+y}(1-R)^{\beta-1} \log (1-R)(-\log R)^n \, dR}{\int_0^1 R^{\alpha+y-1}(1-R)^{\beta-1} (-\log R)^n \, dR}$$

$$- \frac{\int_0^1 R^{\alpha+y}(1-R)^{\beta-1} (-\log R)^n dR \int_0^1 S^{\alpha+y-1}(1-S)^{\beta-1} \log (1-S)(-\log S)^n \, dS}{\left(\int_0^1 R^{\alpha+y-1} (1-R)^{\beta-1} (-\log R)^n \, dR\right)^2}$$

Thus, it suffices to consider

$$\int_0^1 R^{\alpha+y} (1-R)^{\beta-1} \log (1-R)(-\log R)^n \, dR \int_0^1 S^{\alpha+y-1}(1-S)^{\beta-1}(-\log S)^n \, dS$$

$$- \int_0^1 R^{\alpha+y} (1-R)^{\beta-1}(-\log R)^n \, dR \int_0^1 S^{\alpha+y-1}(1-S)^{\beta-1}\log(1-S)(-\log S)^n \, dS$$

$$= \int_0^1 \int_0^1 R^{\alpha+y} (1-R)^{\beta-1}(-\log R)^n S^{\alpha+y}(1-S)^{\beta-1} (-\log S)^n \left[\frac{\log (1-R) - \log (1-S)}{S}\right] dR dS,$$

from which it follows readily that $\partial\hat{R}/\partial\beta < 0$. The verification that $\partial\hat{R}/\partial\alpha > 0$ ($\partial\hat{R}/\partial y > 0$) is completely parallel and is omitted.

3. ASYMPTOTIC COMPARISONS OF ESTIMATORS

From the results of Appendix III (see A.82), we have that

$$\hat{R}_{\alpha,\beta}(L_1) \sim e^{-\frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}(\alpha+\frac{1}{2}) - \frac{(\beta-1)e^{-1/\bar{y}}}{n\bar{y}^2(1-e^{-1/\bar{y}})}}\left(1 - \frac{1}{n\bar{y}}\right), \tag{27}$$

$$\hat{R}_{\alpha,\beta}(L_{2,\nu}) \sim e^{-\frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}(\alpha+\frac{1}{2}) - \frac{(\beta-\nu-1)e^{-1/\bar{y}}}{n\bar{y}^2(1-e^{-1/\bar{y}})}}\left(1 - \frac{1}{n\bar{y}}\right), \tag{28}$$

$$R_{\alpha,\beta}(L_3) \sim e^{-\frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}(\alpha+\frac{1}{2}) - \frac{(\beta-2)e^{-1/\bar{y}}}{n\bar{y}^2(1-e^{-1/\bar{y}})}}\left(1 - \frac{1}{n\bar{y}}\right). \tag{29}$$

As is evident from the above, the differences are small. For large n,

$$\hat{R}_{\alpha,\beta}(L_1)/\hat{R}_{(\alpha,\beta)}(L_{2,\nu}) \sim 1 + \frac{\nu e^{-1/\bar{y}}}{n\bar{y}^2(1-e^{-1/\bar{y}})} \tag{30}$$

and in general $\hat{R}_{(\alpha,\beta)}(L_i)/\hat{R}_{\alpha,\beta}(L_j) \sim 1 + O(n^{-1})$, $1 \le i, j \le 3$.

Naturally, since the maximum likelihood estimator is $e^{-1/\bar{y}}$, one expects the estimators above to be asymptotically equivalent to it. Note further that

$$\nu e^{-1/\bar{y}}/\bar{y}^2(1-e^{-1/\bar{y}})$$

is bounded for all $\bar{y} > 0$.

Similarly, for the "traditional" Bayesian estimator (26), the comparable representation is

$$\hat{R}(\gamma,\delta) \sim e^{-\frac{1}{\bar{y}} - \frac{\gamma}{n\bar{y}} + \frac{2\delta+1}{n\bar{y}^2}}, \tag{31}$$

and $\hat{R}(\gamma,\delta)/\hat{R}_{\alpha,\beta}(L_i) = 1 + O(n^{-1})$, $i = 1,2,3$. Thus, it appears that the differences are about as small as could reasonably be expected.

4. SOME PROPOSED ESTIMATORS OF R

The minimum variance unbiased estimator of R has been studied by E. L. Pugh [8], A. P. Basu [2] and S. Zacks and M. Even [12]. In our notation, this is given by

$$R^* = \begin{cases} (1 - 1/n\bar{y})^{n-1} & n\bar{y} > 1 \\ 0 & n\bar{y} \le 1 \end{cases} \tag{32}$$

The asymptotic representation for R^* is easily seen to be

$$R^* = e^{-\frac{1}{\bar{y}} + \frac{1}{n\bar{y}} - \frac{1}{2n\bar{y}^2}} (1 + O(n^{-2})) .$$

We can also adapt the estimators of the exponential parameter given by G. M. El-Sayyad [4] to correct them in a naive manner to reliability estimators. Naturally, the optimality hypotheses used therein no longer apply,

$$R_1^* = \exp\left(-\frac{\{\Gamma(n-\rho)/\Gamma(n-2\rho)\}^{1/\rho}}{n\bar{y}}\right) \quad , \quad \rho > 0 \tag{33}$$

$$= e^{-\frac{1}{\bar{y}} - (\frac{3\rho+1}{2n\bar{y}})} (1 + O(n-2)) \tag{34}$$

El-Sayyad also provides an argument which leads to the well-known estimator

$$\hat{\theta} = e^{\psi(n)}/n\bar{y} \quad ,$$

where $\psi(n)$ is known as the Psi function or digamma function. Fron this, one deduces

$$\hat{R} = e^{e^{\psi(n)}/n\bar{y}} \tag{35}$$

and from well-known asymptotic properties of $\psi(n)$, (see for example, M. Abramowitz and I. A. Stegun [1]),

$$\hat{R} = e^{\frac{1}{\bar{y}} - \frac{1}{2n\bar{y}}}(1 + O(n^{-2})) \ . \tag{36}$$

The Psi function arises naturally in estimators as the scale transformation invariant estimator for squared error loss. In this connection see T. S. Ferguson [5].

El-Sayyad also obtained some Bayesian point estimators for the exponential parameter using the gamma prior and some loss functions which are generalizations of the squared error loss function.

In the notation of (25), his estimators for θ are

$$\hat{\theta}_1 = (\delta+n\bar{y})^{-1} \{\Gamma(\gamma+n+\alpha+\beta)/\Gamma(\gamma+n+\beta)\}^{1/\beta} \tag{37}$$

and

$$\hat{\theta}_2 = e^{\psi(n+\gamma)}/(\delta+n\bar{y}), \tag{38}$$

where α,β are nonnegative parameters in the loss functions used by El-Sayyad. These yield estimators of R as follows,

$$\hat{R}_i = e^{-\hat{\theta}_i}, \quad i = 1,2 \tag{39}$$

and asymptotically, we have,

$$\hat{R}_1 = e^{-\frac{1}{\bar{y}} - \frac{\gamma+\alpha}{n\bar{y}} - \frac{(\beta-1)}{2n\bar{y}} + \frac{\delta}{n\bar{y}^2}} (1 + O(n^{-2})), \tag{40}$$

and

$$\hat{R}_2 = e^{-\frac{1}{\bar{y}} - \frac{\gamma}{n\bar{y}} + \frac{1}{2n\bar{y}} + \frac{\delta}{n\bar{y}^2}} (1 + O(n^{-2})) \ . \tag{41}$$

Various Bayesian estimators for the exponential parameter were suggested by S. K. Bhattacharyya [3]. In one of these, he considered the range of $\tau = 1/\theta$ to be finite and used the uniform prior

$$g(\tau) = \begin{cases} (\beta-\alpha)^{-1}, & 0 < \alpha \le \tau \le \beta , \\ 0 \quad , & \text{otherwise.} \end{cases} \tag{42}$$

He also considered the inverted gamma density

$$g(\tau) = \frac{e^{-\mu/\tau} (\frac{\mu}{\tau})^{\nu+1}}{\mu \, \Gamma(\nu)} \quad , \quad 0 < \tau < \infty, \quad \mu,\nu > 0. \tag{43}$$

Letting

$$\gamma(n,x) = \int_0^x e^{-t} t^{n-1} dt ,$$

the Bayes estimate for (42) is

$$\hat{R}_3 = \frac{\gamma^*(n-1,n\bar{y}+1)}{\gamma^*(n-1,n\bar{y})} \frac{1}{(1 + 1/n\bar{y})^{n-1}} , \tag{44}$$

where $\gamma^*(n,\nu) = \gamma(n,\frac{\nu}{\alpha}) - \gamma(n, \frac{\nu}{\beta})$.

For the prior density given by (43), the Bayesian estimator was shown to be

$$\hat{R}_4 = (1 + \frac{1}{n\bar{y}+\mu})^{-n-\nu} , \tag{45}$$

which asymptotically behaves like

$$\hat{R}_4 = e^{-\frac{1}{\bar{y}} - \frac{\mu}{n\bar{y}^2} - \frac{1}{2n\bar{y}^2} + \frac{\nu}{n\bar{y}}} (1 + 0(n^{-2})). \tag{46}$$

He also calculated the Bayesian estimator for the exponential

prior obtaining,

$$\hat{R}_5 = \left(K_{n-1}\left(2\sqrt{\frac{n\bar{y}+1}{\lambda}}\right) / K_{n-1}\left(2\sqrt{\frac{n\bar{y}}{\lambda}}\right)\right) \left(1 + \frac{1}{n\bar{y}}\right)^{-\frac{(n-1)}{2}} , \tag{47}$$

where the prior is given by

$$g(\tau) = \lambda^{-1} e^{-\tau/\lambda} , \qquad 0 < \tau < \infty , \quad \lambda > 0 . \tag{48}$$

It can be shown that (47) may be approximated by

$$\hat{R}_5 \sim \left(1 + \frac{1}{n\bar{y}}\right)^{-(n-1)} \tag{49}$$

for large n. Formula 9.7.8 p. 378 in M. Abramowitz and I. A. Stegun [1] may be used to obtain more detailed asymptotic information.

In the papers by S. K. Sinha and I. Guttman [10], [11] the improper prior

$$g(\tau) = \tau^{-1}, \quad 0 < \tau < \infty \tag{50}$$

is assigned to $\tau = 1/\theta$, obtaining as the Bayes estimator

$$\hat{R} = (1 + 1/n\bar{y})^{-n} , \tag{51}$$

which yields the asymptotic expression

$$\hat{R} = e^{-\frac{1}{\bar{y}} - \frac{1}{2n\bar{y}^2}} (1 + O(n^{-2})) . \tag{52}$$

V. M. Rao Tummala and P. T. Sathe [9] employed gamma priors and obtained "minimum expected loss estimators" for the reliability. They compare these estimators with the Bayes and maximum likelihood estimators for quadratic loss functions in estimating θ .

APPENDIX I

In this appendix, we study the series given in (9) and establish the remarks following (9) and (10). This can also serve as a justification for the interchange of operations used in deriving (9). Specifically, we establish the following

<u>Theorem A.1</u>. Let

$$I(\alpha,\beta,\tau) = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} (\alpha+i)^{-(\tau+1)}, \tag{A.1}$$

where $\alpha > 0$, and τ and β are real, and where

$$\binom{\beta-1}{i} = \begin{cases} (\beta-1)(\beta-2)\cdots(\beta-i)/i! & i = 1,2,\ldots \\ 1\,, & i = 0\,. \end{cases} \tag{A.2}$$

Then $I(\alpha,\beta,\tau)$ converges whenever β is a positive integer. If β is not a positive integer, $I(\alpha,\beta,\tau)$ converges if and only if $\beta + \tau > 0$.

<u>Proof</u>. Clearly, if β is a positive integer, the series (A.1) terminates and convergence is trivially verified. Hence assume that β is not a positive integer. Write

$$\begin{aligned} I(\alpha,\beta,\tau) = & \sum_{i<\beta} \binom{\beta-1}{i}(-1)^i(\alpha+i)^{-(\tau+1)} \\ & + \sum_{i\geq\beta} \binom{\beta-1}{i}(-1)^i(\alpha+i)^{-(\tau+1)}\,. \end{aligned} \tag{A.3}$$

The first sum has a finite (possibly zero) number of terms. Hence, to study convergence, it suffices to restrict attention to the second sum, which we

write as

$$\sum_{i \geq \beta} C_\beta \frac{\gamma(\gamma+1)\cdots(\gamma+i-r)(\alpha+i)^{-(\tau+1)}}{i!} , \tag{A.4}$$

where

$$r = 1, \quad C_\beta = 1, \ \gamma = 1-\beta, \quad \text{if} \quad \beta \leq 0$$

$$r = [\beta]+1, \quad C_\beta = (-1)^{r-1}(\beta-1)\cdots(\beta-r+1), \ \gamma = (r-\beta) \quad \text{if} \quad \beta > 0 . \tag{A.5}$$

Further, in (A.4), vacuous products are interpreted as unity and γ is always positive.

We now determine when (A.4) converges and obtain estimates for the tail of (A.4), when it is convergent. Hence, it suffices to consider

$$I_M(\alpha,\gamma,r,\tau) = \sum_{i \geq M} \frac{\gamma(\gamma+1)\cdots(\gamma+i-r)(\alpha+i)^{-(\tau+1)}}{i!} , \tag{A.6}$$

which has all terms positive. Note that the sign of (A.4) is determined solely by C_β.

Now

$$\log\left[\left(\prod_{j=0}^{i-r} (\gamma+j)\right) (\alpha+i)^{-(\tau+1)}/i!\right] \leq$$

$$(\gamma+i-r)\log(\gamma+i-r)-(\gamma+i-r)-\gamma\log\gamma+\gamma+\frac{1}{2}\log[(\gamma)(\gamma+i-r)]$$

$$-(\tau+1)\log(\alpha+i) - i\log i + i - \frac{1}{2}\log i - \frac{1}{2}\log(2\Pi)$$

$$=(\gamma+i-r)[\log i + \log(1+\frac{\gamma-r}{i})] + r - \gamma\log\gamma + \frac{1}{2}\log\gamma + \frac{1}{2}\log i$$

$$+\frac{1}{2}\log(1+\frac{\gamma-r}{i})-(\tau+1)\log i - (\tau+1)\log(1+\frac{\alpha}{i}) - i\log i$$

$$-\frac{1}{2}\log i - \frac{1}{2}\log(2\Pi). \tag{A.7}$$

For M sufficiently large, we have for all $i \geq M$,

$$\frac{1}{2} < \left(1 + \frac{\gamma - r}{i}\right) \leq e^{\frac{\gamma - r}{i}} < e$$

and

$$1 < \left(1 + \frac{\alpha}{i}\right) < \frac{3}{2} \quad .$$

Thus,

$$\left(\prod_{j=0}^{i-r} (\gamma + j)\right) (\alpha + i)^{-(\tau+1)}/i! \leq d(\tau)\left(\frac{\gamma}{2\pi}\right)^{\frac{1}{2}} \gamma^{-\gamma} e^{2\gamma - r + \frac{1}{2}} \, i^{\gamma - r - \tau - 1} \quad , \tag{A.8}$$

where

$$d(\tau) = \begin{cases} 1, & \tau \geq -1 \\ \left(\frac{3}{2}\right)^{-(\tau+1)}, & \tau < -1 \end{cases} \quad . \tag{A.9}$$

Therefore, we have shown that

$$I_M \leq c \sum_{i \geq M} i^{\gamma - r - \tau - 1} \quad , \tag{A.10}$$

for some positive constant $c = c(\gamma, r, \tau)$. From (A.10), we can immediately observe that I_M converges absolutely whenever $\gamma - r - \tau < 0$. Thus, from (A.5), we have that if $\beta > 0$, $I(\alpha, \beta, \tau)$ converges whenever $\beta + \tau > 0$. Similarly, if $\beta \leq 0$, the same conclusion holds.

To establish the converse, note that

$$\log \left[\left(\prod_{j=0}^{i-\gamma} (\gamma + j)\right) (\alpha + i)^{-(\tau+1)}/i!\right] \geq$$

$$(\gamma+i-r)[\log i + \log(1 + \frac{\gamma-r}{i})] + r - \gamma \log \gamma + \frac{1}{2} \log \gamma + \frac{1}{2} \log i$$

$$+ \frac{1}{2} \log (1 + \frac{\gamma-r}{i}) + \frac{1}{12}[(\gamma+i-r)^{-1} - \gamma^{-1}] - (\tau+1) \log(1 + \frac{\alpha}{i})$$

$$-(\tau+1) \log i - i \log i - \frac{1}{2} \log i - \frac{1}{2} \log (2\Pi) - \frac{1}{12i} \cdot$$

Now, for M sufficiently large and all $i > M$

$$(\gamma+i-r)^{-1} > 0 \quad ,$$

$$(12i)^{-1} \leq 1/12$$

and

$$(1 + \frac{\gamma-r}{i}) \geq e^{(\gamma-r)/(i+(\gamma-r))} \quad .$$

Consequently,

$$\left(\prod_{j=0}^{i-r} (\gamma+j) \right) (\alpha+i)^{-(\tau+1)}/i! \geq$$

$$(2\pi)^{-\frac{1}{2}} e^{\gamma - \frac{1}{12} - \frac{1}{12\gamma}} i^{\gamma-r-\tau-1} \gamma^{-\gamma+\frac{1}{12}} d_1(\tau) \quad , \tag{A.11}$$

where $d_1(\tau)$ is defined by (A.9).

Thus

$$I_M \geq k \sum_{i \geq M} i^{\gamma-r-\tau-1} \quad , \tag{A.12}$$

where $k = k(\gamma,\tau,r)$ is a positive constant. Thus I_M diverges whenever $\gamma-r-\tau \geq 0$ and thus diverges whenever $\beta+\tau \leq 0$, establishing the theorem.

Remark. The above analysis permits us to readily estimate the tail of the series whenever $\beta+\tau > 0$, since

$$\sum_{i \geq M} i^{-(\beta+\tau+1)} < \int_{M-1}^{\infty} x^{-(\beta+\tau+1)} dx = \frac{(M-1)^{-(\beta+\tau)}}{(\beta+\tau)} \quad . \tag{A.13}$$

APPENDIX II

In this appendix, we obtain a complete asymptotic expansion $(n\to\infty)$ of

$$I_n(\alpha,\beta,y) = \int_0^1 R^{\alpha+y-1}(1-R)^{\beta-1}(-\log R)^n dR \quad ,$$

where $y = n\bar{y}$, $\bar{y} > 0$. In particular, we establish the following theorem.

Theorem A.2. For $\bar{y} > 0$, $\alpha > 0$, as $n \to \infty$, for any $\delta > 0$, $k = 2,3,\ldots$

$$e^{M_n(\theta_n^*)} I_n(\alpha,\beta,y) = \sqrt{2\pi/\gamma_n}\Big(\sum_{r=0}^{k} K_{n,k}^{(r)}(\theta_n^*)\mu_r/\gamma_n^{r/2}\Big) + O\left(n^{-\frac{k+1}{6}-\frac{1}{2}+\delta}\right) \quad , \tag{A.14}$$

where

$$M_n(\theta) = \theta(\alpha+n\bar{y}) - n\log\theta - (\beta-1)\log(1-e^{-\theta}) \tag{A.15}$$

and θ_n^* is the largest positive root of $M_n'(\theta) = 0$. Further,

$$\gamma_n = n\left[\frac{1}{\theta_n^{*2}} - \frac{\beta-1}{n}\left(\left(1-e^{-\theta_n^*}\right)^{-2} - \left(1-e^{-\theta_n^*}\right)^{-1}\right)\right] \quad , \tag{A.16}$$

$$\mu_r = \begin{cases} r!/(\frac{r}{2})!\ 2^{r/2}, & r \text{ even} \\ 0 \quad , & r \text{ odd} \end{cases} \tag{A.17}$$

and

$$K_{nk}^{(r)}(\theta) = \Sigma(-1)^{\Sigma\ell_i} M_n^{(3)}(\theta)^{\ell_3} \cdots\cdots M_n^{(k)}(\theta)^{\ell_k}/\ell_3!\cdots\ell_k!\ 3!^{\ell_3}\cdots k!^{\ell_k} \quad , \tag{A.18}$$

where the sum extends over all $\ell_3,\ldots,\ell_k \geq 0$ with $\sum_{j=3}^{k} j\ell_j = r$.

In order to prove Theorem A.2, it is desirable to introduce a number of preliminary lemmas.

Lemma A.1. For $j = 1,2,\ldots,$

$$M_n^{(j)}(\theta) = (\alpha+n\bar{y})\delta_{1j} + \frac{(-1)^j(j-1)!n}{\theta^j} - (\beta-1)\sum_{\ell=0}^{j} b_{\ell j}\left(1-e^{-\theta}\right)^{-\ell} \quad , \tag{A.19}$$

where $\delta_{11} = 1$, $\delta_{1j} = 0$, $j \neq 1$ and

$$\begin{cases} b_{01} = -1,\ b_{11} = 1,\ b_{\ell j} = 0,\ \ell > j \quad \text{or} \quad \ell < 0 \ , \\ b_{\ell j} = \ell b_{\ell,j-1} - (\ell-1)b_{\ell-1,j-1},\ 0 \le \ell \le j,\ j \ge 2 \ . \end{cases} \tag{A.20}$$

Alternatively, we can write

$$M_n^{(j)}(\theta) = (\alpha+n\bar{y})\delta_{1j} + \frac{(-1)^j(j-1)!n}{\theta^j} - (\beta-1)\sum_{\ell=1}^{j} a_{\ell j}e^{-\ell\theta}\left(1-e^{-\theta}\right)^{-j} \ , \tag{A.21}$$

where

$$\begin{cases} a_{11} = 1,\ a_{0j} = 0,\ a_{jj} = 0,\ j > 1,\ a_{\ell j} = 0,\ \ell > j \\ a_{\ell,j+1} = -\ell a_{\ell j} - (j-\ell+1)a_{\ell-1,j} \ ,\ 1 \le \ell \le j,\ j \ge 2 \ . \end{cases} \tag{A.22}$$

Further, for $j \ge 2$

$$\frac{n(j-1)!}{\theta^j} - |\beta-1|e^{-\theta}(1-e^{-\theta})^{-j}(j-1)! \le |M_n^{(j)}(\theta)| \le \frac{n(j-1)!}{\theta^j}$$

$$+ |\beta-1|e^{-\theta}(1-e^{-\theta})^{-j}(j-1)! \tag{A.23}$$

and for n sufficiently large, for each fixed θ

$$\frac{n(j-1)!}{2\theta^j} \le |M_n^{(j)}(\theta)| \le \frac{2n(j-1)!}{\theta^j} \ . \tag{A.24}$$

<u>Proof</u>. (A.19), (A.20), (A.21) and (A.22) are easily verified. From (A.22), it is evident that $\operatorname{sgn} a_{\ell j} = (-1)^{j+1}$, $j \ge 1$, $1 \le \ell \le j$. Further

$$\sum_{\ell=1}^{j} a_{\ell,j+1} = \sum_{\ell=1}^{j} (-\ell a_{\ell j}) - \sum_{\ell=1}^{j} (j-\ell+1)a_{\ell-1,j}$$

$$= -\sum_{\ell=0}^{j} [\ell a_{\ell j} + (j-\ell)a_{\ell j}] = -j \sum_{\ell=0}^{j} a_{\ell j} \ .$$

Thus

$$\sum_{\ell=0}^{j} a_{\ell j} = (-1)^{j+1}c(j-1)!$$

for some constant c. Since $a_{11} = 1$, it follows that $c = 1$, and

$$\sum_{\ell=0}^{j} a_{\ell j} = (-1)^{j+1}(j-1)! \ . \tag{A.25}$$

Thus, from (A.25), we have

$$\left|\sum_{\ell=1}^{j} a_{\ell j} e^{-\ell\theta}(1-e^{-\theta})^{-j}\right| \le e^{-\theta}(1-e^{-\theta})^{-j}(j-1)! \quad . \tag{A.26}$$

(A.23 and (A.24) follow directly from (A.26), establishing the lemma.

Lemma A.2. A sequence of asymptotic $(n\to\infty)$ estimates of θ_n^* is provided by

$$\theta_n^* = \bar{\theta}_i + O(n^{-i}), \; i = 1,2,\ldots \tag{A.27}$$

where

$$\bar{\theta}_1 = 1/\bar{y} \quad , \tag{A.28}$$

and

$$\bar{\theta}_{i+1} = \bar{\theta}_i + \frac{\bar{\theta}_i^2}{n}\left(\frac{n}{\bar{\theta}_i} + \frac{\beta-1}{1-e^{-\bar{\theta}_i}} - n\bar{y} - (\alpha+\beta-1)\right) \quad . \tag{A.29}$$

Proof. For every $\varepsilon > 0$, there is an n sufficiently large so that $M_n(\theta)$ is strictly convex on (ε,∞). Thus, $M_n'(\theta) = 0$ has a unique root θ_n^* on (ε,∞). (A.27), (A.28) and (A.29) follow readily.

Specifically,

$$\begin{aligned}
\theta_n^* &= \frac{1}{\bar{y}} + O(n^{-1}) \\
&= \frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right) + O(n^{-2}) \\
&= \frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right) + \frac{1}{n^2\bar{y}^3}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right)^2 \\
&\quad - \frac{(\beta-1)e^{-1/\bar{y}}}{n^2\bar{y}^4(1-e^{-1/\bar{y}})^2}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right) + O(n^{-3}) \quad .
\end{aligned} \tag{A.30}$$

Remark: Applying lemma A.2, and letting $\theta = \theta_n^*$ in (A.23), we can deduce

$$n(j-1)!/2\theta_n^{*j} \le |M_n^{(j)}(\theta_n^*)| \le 2n(j-1)!/\theta_n^{*j} \quad . \tag{A.31}$$

Lemma A.3. For $0 \le r \le k$, $k \ge 2$,

$$K_{nk}^{(0)}(\theta_n^*) = 1 \tag{A.32}$$

$$K_{nk}^{(r)}(\theta_n^*) = 0, \ r = 1,2 \tag{A.33}$$

$$|K_{nk}^{(r)}(\theta_n^*)| \le k(2n)^{r/3}\,\bar{y}^{-r}, \ r = 3,4,\ldots,k \ , \tag{A.34}$$

for n sufficiently large.

<u>Proof</u>. From (A.18), (A.32) and (A.33) follows trivially. Therefore, we need only establish (A.34). From (A.24) and (A.18), for n sufficiently large, $3 \le r \le k$,

$$|K_{nk}^{(r)}(\theta)| \le \Sigma\Big(\frac{2n(2!)}{\theta^3}\Big)^{\ell_3}\cdots\Big(\frac{2n(k-1)!}{\theta^k}\Big)^{\ell_k}/\ell_3!\cdots\ell_k!\ 3!^{\ell_3}\cdots k!^{\ell_k}$$

$$= \Sigma\Big(\frac{(2n)^{\Sigma\ell_n}}{\theta^{\Sigma j\ell_j}}/\ell_3!\cdots\ell_k!\ 3^{\ell_3}\cdots k^{\ell_k} \ . \tag{A.35}$$

Since

$$\sum_{j=3}^{k} j\ell_j = r \ ,$$

we have

$$3\sum_{j=3}^{k} \ell_j \le r$$

and

$$\sum_{j=3}^{k} \ell_j \le r/3 \ .$$

Thus

$$|K_{nk}^{(r)}(\theta)| \le \frac{(2n)^{r/3}}{\theta^r}\,\Sigma 1/\ell_3!\cdots\ell_k!\ 3^{\ell_3}\cdots k^{\ell_n} \ .$$

Now

$$\Sigma 1/\ell_3!\cdots\ell_k!\ 3^{\ell_3}\cdots k^{\ell_k} \le \sum_{m=1}^{[r/3]} \frac{1}{m!} \sum_{\ell_3,\ldots,\ell_k \ge 0} \frac{m!}{\ell_3!\cdots\ell_k!}\Big(\frac{1}{3}\Big)^{\ell_3}\cdots\Big(\frac{1}{k}\Big)^{\ell_k}$$

$$= \sum_{m=1}^{[r/3]} \frac{1}{m!}\Big(\frac{1}{3}+\cdots+\frac{1}{k}\Big)^m$$

$$\le \sum_{m=1}^{\infty} \frac{1}{m!}(\log k)^m = k \ .$$

Thus

$$|K_{nk}^{(r)}(\theta)| < k(2n)^{r/3}/\theta^r \quad .$$

From (A.31), the conclusion follows readily upon replacing θ by θ_n^* in (A.35).

We now proceed to the proof of the theorem.

<u>Proof</u>. Let $R = e^{-\theta}$ obtaining

$$I_n(\alpha,\beta,y) = \int_0^\infty e^{-\theta(\alpha+n\bar{y})}\left(1-e^{-\theta}\right)^{\beta-1}\theta^n d\theta \quad . \tag{A.36}$$

Let

$$g(\theta) = e^{-\theta(\alpha+n\bar{y})}\left(1-e^{-\theta}\right)^{\beta-1}\theta^n \quad . \tag{A.37}$$

Choose $\tau_1(n)$, $\tau_2(n)$ so that

$$0 < \tau_1(n) < n/(\alpha+n\bar{y}) < \tau_2(n) < \infty \quad . \tag{A.38}$$

For $\beta > 1$,

$$\int_0^{\tau_1(n)} g(\theta)d\theta < \int_0^{\tau_1(n)} e^{-\theta(\alpha+n\bar{y})}\theta^n d\theta < (\tau_1(n))^{n+1}e^{-\tau_1(n)(\alpha+n\bar{y})} \quad . \tag{A.39}$$

For $\beta < 1$, since $1-e^{-\theta} > \theta(1-e^{-\tau_1})/\tau_1$, $0 < \theta < \tau_1$,

$$\int_0^{\tau_1(n)} g(\theta)d\theta < \left(\frac{1-e^{\tau_1(n)}}{\tau_1(n)}\right)^{\beta-1} \int_0^{\tau_1(n)} e^{-\theta(\alpha+n\bar{y})}\theta^{n+\beta-1}d\theta \quad ,$$

whenever

$$\tau_1(n) < \frac{n+\beta-1}{\alpha+n\bar{y}} \quad ,$$

we have

$$\int_0^{\tau_1(n)} g(\theta)d\theta < \left(\frac{1-e^{-\tau_1(n)}}{\tau_1(n)}\right) \tau_1(n)^{n+\beta}\, e^{-\tau_1(n)(\alpha+n\bar{y})} \quad . \tag{A.40}$$

Proceeding similarly, we have,

$$\int_{\tau_2(n)}^\infty g(\theta)d\theta < f(\beta) \int_{\tau_2(n)}^\infty e^{-\theta(\alpha+ny)}\theta^n d\theta \quad ,$$

where

$$f(\beta) = 1 \quad , \quad \beta > 1$$

$$f(\beta) = \left(1-e^{-\tau_2(n)}\right)^{\beta-1} , \ \beta < 1 \quad .$$

From F. W. J. Olver [7], p. 70, we have

$$\int_{\tau_2(n)}^{\infty} e^{-\theta(\alpha+n\bar{y})}\theta^n d\theta = (\alpha+n\bar{y})^{-n-1} \int_{(\alpha+n\bar{y})\tau_2(n)}^{\infty} e^{-u}u^n du$$

$$< \frac{e^{-(\alpha+n\bar{y})\tau_2(n)}(\tau_2(n))^{n+1}}{(\alpha+n\bar{y})\tau_2(n)-n} .$$

Hence

$$\int_{\tau_2(n)}^{\infty} g(\theta)d\theta < \frac{f(\beta)e^{-(\alpha+n\bar{y})\tau_2(n)}(\tau_2(n))^{n+1}}{(\alpha+n\bar{y})\tau_2(n)-n} . \tag{A.41}$$

From (A.15) and (A.36), we have that

$$I_n(\alpha,\beta,y) = \int_0^{\infty} e^{-M_n(\theta)} d\theta . \tag{A.42}$$

Now write, for $k > 2$

$$M_n(\theta) = M_n(\theta_n^*) + \frac{(\theta-\theta_n^*)^2}{2!} M_n''(\theta_n^*) + \cdots + \frac{(\theta-\theta_n^*)^k}{k!} M_n^{(k)}\theta_n^*$$
$$+ W_{nk}(\theta) , \tag{A.43}$$

where

$$|W_{nk}(\theta)| < \frac{|\theta-\theta_n^*|^{k+1}}{(k+1)!} |M_n^{(k+1)}(\xi_n)| \tag{A.44}$$

and ξ_n lies between θ and θ_n^*.

From (A.30) or (A.15), it is easily seen that $\theta^* = n/(\alpha+n\bar{y}) + o(1)$ and thus for sufficiently large n,

$$\tau_1(n) < \theta_n^* < \tau_2(n) .$$

Consequently for $\theta \in (\tau_1(n), \tau_2(n))$, from (A.24) and (A.44),

$$|W_{nk}(\theta)| < \frac{(\tau_2(n)-\tau_1(n))^{k+1}}{(k+1)!} \left(\frac{2nk!}{\xi_n^{k+1}}\right) . \tag{A..45}$$

Now write $g(\theta)$ as

$$g(\theta) = e^{-M_n(\theta_n^*) - \frac{\gamma_n}{2}(\theta-\theta_n^*)^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\sum_{\ell=3}^{k} \frac{(\theta-\theta_n^*)^\ell}{\ell!} M_n^{(\ell)}(\theta_n^*)\right)^r \cdot$$

$$\cdot e^{-W_{nk}(\theta)} , \tag{A.46}$$

where $\gamma_n = M_n''(\theta_n^*)$. Then write

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\sum_{\ell=3}^{k} \frac{(\theta-\theta_n^*)^\ell}{\ell!} M_n^{(\ell)}(\theta_n^*) \right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \sum_{\ell_3,\ldots,\ell_k \geq 0} \frac{r!}{\ell_3!\ldots\ell_k!} \frac{(\theta-\theta_n^*)^{3\ell_3+\ldots+k\ell_k} M_n^{(3)}(\theta_n^*)^{\ell_3} \ldots M_n^{(k)}(\theta_n^*)^{\ell_k}}{3!^{\ell_3}\ldots k!^{\ell_k}}$$

$$= 1 + \sum_{\ell=3}^{\infty} \sum_{\ell_3,\ldots,\ell_k} \frac{(-1)^{\sum_3^k \ell_j} (\theta-\theta_n^*)^\ell \prod_{j=3}^{k} \left(M_n^{(j)}(\theta_n^*)\right)^{\ell_j}}{\prod_{j=3}^{k} \ell_j! \prod_{j=3}^{k} j!^{\ell_j}} , \qquad \text{(A.47)}$$

where the sum is over $\ell_3,\ldots,\ell_k \geq 0$ with $\Sigma j\ell_j = \ell$. Thus, we can write (A.47) as

$$1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}(\theta_n^*)(\theta-\theta_n^*)^\ell + \sum_{\ell=k+1}^{\infty} K_{nk}^{(\ell)}(\theta_n^*)(\theta-\theta_n^*)^\ell , \qquad \text{(A.48)}$$

where

$$K_{nk}^{(\ell)} = \sum_{\ell_3,\ldots,\ell_k} (-1)^{\Sigma \ell j} \prod_{j=3}^{k} \left(M_n^{(j)}(\theta_n^*)\right)^{\ell_j} / \prod_{j=3}^{k} \ell_j! \prod_{j=3}^{k} j!^{\ell_j} . \qquad \text{(A.49)}$$

Accordingly, we define

$$h_k(\theta) = e^{-M_n(\theta_n^*) - \frac{\gamma_n}{2}(\theta-\theta_n^*)^2} \left(1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)} (\theta-\theta_n^*)^\ell\right) , \qquad \text{(A.50)}$$

and consider

$$e^{M_n(\theta_n^*)} \left| \int_0^\infty g(\theta)d\theta - \int_{-\infty}^\infty h_k(\theta)d\theta \right| = R_k . \qquad \text{(A.51)}$$

Then

$$R_k \leq e^{M_n(\theta_n^*)}\{\int_{-\infty}^{\tau_1(n)} h_k(\theta)d\theta + \int_0^{\tau_1(n)} g(\theta)d\theta + \int_{\tau_1(n)}^{\tau_2(n)} |h_k(\theta)-g(\theta)|d\theta$$

$$+ \int_{\tau_2(n)}^{\infty} g(\theta)d\theta + \int_{\tau_2(n)}^{\infty} h_k(\theta)d\theta\} \quad . \qquad \text{(A.52)}$$

From (A.34), we have

$$|1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}(\theta-\theta_n^*)^\ell| \leq 1 + k \sum_{\ell=3}^{k} (2n)^{\ell/3}\, \bar{y}^{\ell}(\theta-\theta_n^*)^\ell \quad .$$

Thus for $\tau_2(n) > \theta_n^*$,

$$e^{M_n(\theta_n^*)} \int_{\tau_2(n)}^{\infty} h_k(\theta)d\theta = \int_{\tau_2(n)}^{\infty} e^{-\frac{\gamma_n}{2}(\theta-\theta_n^*)^2}\left(1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}(\theta-\theta_n^*)^\ell\right)d\theta \quad .$$

Setting $w = \frac{\gamma_n}{2}(\theta-\theta_n^*)^2$, we obtain

$$e^{M_n(\theta_n^*)} \int_{\tau_2(n)}^{\infty} h_k(\theta)d\theta = \frac{1}{\sqrt{2\gamma_n}} \int_{\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2}^{\infty} e^{-w}\left(w^{-\frac{1}{2}} + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}\left(\frac{2}{\gamma_n}\right)^{\ell/2} w^{\frac{\ell-1}{2}}\right)dw \quad .$$

Then, applying (A.31) with $j = 2$, and (A.41)

$$\left|\frac{1}{\sqrt{2\gamma_n}} \int_{\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2}^{\infty} e^{-w}(w^{-\frac{1}{2}} + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}\left(\frac{2}{\gamma_n}\right)^{\ell/2} w^{\frac{\ell-1}{2}} dw\right|$$

$$\leq \frac{1}{n^{1/2}\bar{y}} \int_{\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2}^{\infty} e^{-w}\left(w^{-\frac{1}{2}} + k \sum_{\ell=3}^{k} \frac{2^{4\ell/3}}{n^{\ell/6}}\, w^{\frac{\ell-1}{2}}\right)dw$$

$$\leq \frac{1}{n^{1/2}\bar{y}}\left[e^{-\frac{\gamma_n}{2}(\tau_2-\theta_n^*)^2}\left(\left[\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2\right]^{-\frac{1}{2}}\right.\right.$$

$$+ k \sum_{\ell=3}^{k} \frac{2^{4\ell/3}}{n^{\ell/6}} \frac{\left[\gamma_n(\tau_2(n)-\theta_n^*)^2\right]^{\frac{\ell+1}{2}}}{\left[\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2 - \frac{\ell-1}{2}\right]}\Bigg)\Bigg] \quad ,$$

the last inequality following from inequalities for the incomplete gamma function on pages 66 and 70 of F. J. Olver [7].

Now let $n^{-\frac{1}{3}+\delta} < \tau_2(n)-\theta_n^* = \theta_n^* - \tau_1(n) < n^{-\frac{1}{3}-\delta}$, $0 < \delta < \frac{1}{12}$. Then for n sufficiently large, $3 \leq \ell \leq k$,

$$\frac{k2^{4\ell/3}}{n^{\ell/6}}\left(\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2\right)^{(\ell+1)/2} \Bigg/ \left(\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2 - (\ell-1)/2\right)$$

$$\leq \frac{2k2^{4\ell/3}}{n^{\ell/6}}\left(\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2\right)^{(\ell+1)/2}\left(\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2\right)$$

$$= \frac{2^{(4\ell+3)/3}k}{n^{\ell/6}}\left(\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2\right)^{\frac{\ell-1}{2}}$$

$$\leq kn^{-\ell/6}2^{(4\ell+3)/3}\left[(n\bar{y}^2)n^{-\frac{2}{3}}\right]^{\frac{\ell-1}{2}} = k2^{(4\ell+3)/3}n^{-1/6} \quad . \tag{A.53}$$

Similarly,

$$\left[\frac{\gamma_n}{2}(\tau_2(n)-\theta_n^*)^2\right]^{-\frac{1}{2}} \leq \left[\frac{n\bar{y}^2}{4}n^{-1+2\delta}\right]^{-\frac{1}{2}}$$

$$= n^{-\delta}\bar{y}^2/4 \quad .$$

Consequently, there is a constant $c_{k,\bar{y}}$ such that

$$\left|e^{M_n(\theta_n^*)}\int_{\tau_2(n)}^{\infty} h_k(\theta)d\theta\right| \leq \frac{c_{k,\bar{y}}\, e^{-n^{2\delta}\bar{y}^2/4}}{n^{\frac{1}{2}+\delta}} \quad . \tag{A.54}$$

Similar calculations establish the same bound for

$$e^{M_n(\theta_n^*)} \int_0^{\tau_1(n)} h_k(\theta)d\theta \quad .$$

We now consider $e^{M_n(\theta_n^*)} \int_{\tau_1(n)}^{\tau_2(n)} (h_k(\theta)-g(\theta))d\theta$. From (A.46) and (A.50),

$$e^{M_n(\theta_n^*)}(g(\theta) - h_k(\theta)) = e^{-\frac{\gamma_n}{2}(\theta-\theta_n^*)^2} \Big((1 + \sum_{\ell+3}^{k} K_{nk}^{(\ell)}(\theta-\theta_n^*)^{\ell}\Big) \quad .$$

$$(e^{-W_{nk}(\theta)} - 1) + e^{-W_{nk}(\theta)} \sum_{\ell=k+1}^{\infty} K_{nk}^{(\ell)}(\theta-\theta_n^*)^{\ell} \quad . \qquad \text{(A.55)}$$

From (A.44) and (A.31), for sufficiently large n,

$$|W_{nk}(\theta)| \leq \frac{\big(\tau_2(n)-\tau_1(n)\big)^{k+1}}{(k+1)!} \, 4nk!\bar{y} \quad ,$$

since $|\xi_n - \frac{1}{\bar{y}}| \leq \tau_2(n) - \tau_1(n) \leq 2n^{-1/3}$. Thus

$$|e^{-W_{nk}(\theta)} - 1| = O\big((\tau_2(n)-\tau_1(n))^{k+1}n\big)$$

and

$$|W_{nk}| = O((\tau_2(n) - \tau_1(n))^{k+1}n)$$

as $n \to \infty$. Further from (A.34),

$$|\sum_{\ell=k+1}^{\infty} K_{nk}^{(\ell)}(\theta-\theta_n^*)^{\ell}| \leq k \sum_{\ell=k+1}^{\infty} (2n)^{\ell/3}\bar{y}^{\ell}(\tau_2(n) - \tau_1(n))^{\ell}$$

$$= \frac{k(2n)^{(k+1)/3} \, \bar{y}^{(k+1)}(\tau_2(n) - \tau_1(n))^{(k+1)}}{1 - (2n)^{1/3}\bar{y}(\tau_2(n) - \tau_1(n))} \quad . \qquad \text{(A.56)}$$

Since $n(\tau_2(n) - \tau_1(n))^3 \to 0$ as $n \to \infty$,

$$|\sum_{\ell=k+1}^{\infty} K_{nk}^{(\ell)}(\theta-\theta_n^*)^{\ell}| \leq 2k(2n)^{(k+1)/3} \, \bar{y}^{(k+1)}(\tau_2(n) - \tau_1(n))^{k+1} \qquad \text{(A.57)}$$

for sufficiently large n. Similarly

$$|1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}(\theta-\theta_n^*)^{\ell}|$$

$$\leq 1 + \sum_{\ell=3}^{k} k(2n)^{\ell/3-\ell}\bar{y}^{\ell}(\tau_2(n) - \tau_1(n))^{\ell}$$

$$= 1 + O\left(\tau_2(n) - \tau_1(n))^{\ell} n^{\ell/3}\right) \quad . \tag{A.58}$$

Thus

$$\left|1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}(\theta-\theta_n^*)^{\ell}\right|\left(O(\tau_2(n) - \tau_1(n)^{k+1}n\right)$$

$$= O\left((\tau_2(n) - \tau_1(n)^{k+1}n\right) \quad . \tag{A.59}$$

Hence

$$\int_{\tau_1(n)}^{\tau_2(n)} |g(\theta) - h(\theta)|d\theta \leq \int_{\tau_1(n)}^{\tau_2(n)} e^{-\frac{\gamma_n}{2}(\theta-\theta_n^*)^2}\left[2k(2n)^{(k+1)/3}\,\bar{y}^{\,k+1}\right. \cdot$$

$$\left. \cdot\ (\tau_2(n) - \tau_1(n))^{k+1} + O(\tau_2-\tau_1)^{k+1}n)\right] d\theta$$

$$= O\left((\tau_2(n)-\tau_1(n))^{k+1}n^{(k+1)/3}\right)\int_{\tau_1(n)}^{\tau_2(n)} e^{-\frac{\gamma_n}{2}(\theta-\theta_n^*)^2} d\theta$$

$$= O\left((\tau_2(n)-\tau_1(n))^{k+1}n^{(k+1)/3}\right) \frac{1}{\sqrt{\gamma_n}} \int_{\tau_1(n)\sqrt{\gamma_n}}^{\tau_2(n)\sqrt{\gamma_n}} e^{-(\theta-\theta_n^*)^2/2}d\theta$$

$$= O\left((\tau_2(n)-\tau_1(n))^{(k+1)}n^{(k+1)/3-\frac{1}{2}}\right)$$

$$= O\left(n^{-\frac{k+1}{6}-\frac{1}{2}+(k+1)\delta}\right) \quad . \tag{A.60}$$

From (A.37) and (A.40)

$$\int_0^{\tau_1(n)} g(\theta)d\theta \leq (\tau_1(n))^{n+d_\beta}\, e^{-(\alpha+n\bar{y})\tau_1(n)}\, r(\tau_1(n)) \quad ,$$

where $d_\beta = \begin{cases} 1 & , \quad \beta > 1 \\ \beta & , \quad \beta < 1 \end{cases}$

and

$$r(\tau_1(n) = \begin{cases} 1 & \beta > 1 \\ \left(1-e^{-\tau_1(n)}\right)/\tau_1(n), & \beta < 1 \end{cases} .$$

Thus

$$\int_0^{\tau_1(n)} g(\theta)d\theta/e^{-M_n(\theta_n^*)}$$

$$= \left(\frac{\tau_1(n)}{\theta^*(n)}\right)^n e^{(\theta_n^*-\tau_1(n))(\alpha+n\bar{y})} (\tau_1(n))^{d_\beta} \frac{r(\tau_1(n))}{\left(1-e^{-\theta_n^*}\right)^{\beta-1}} .$$

Since

$$\frac{1}{2\bar{y}} < \tau_1(n) < \frac{2}{\bar{y}} \tag{A.61}$$

for n sufficiently large, we can write

$$(\tau_1(n))^{d_\beta} r(\tau_1(n))/\left(1-e^{-\theta_n^*}\right)^{\beta-1} < d_1(\bar{y},\beta) , \tag{A.62}$$

where $d_1(\bar{y},\beta)$ does not depend on n. Similarly,

$$\int_{\tau_2(n)}^\infty g(\theta)d\theta < \frac{f_1(\beta)e^{-(\alpha+n\bar{y})\tau_2(n)}}{(\alpha+n\bar{y})\tau_2(n)-n} , \tag{A.63}$$

where

$$f_1(\beta) = \tau_2(n)f(\beta) ,$$

and from (A.61), for n sufficiently large

$$f_1(\beta) < d_2(\bar{y},\beta) ,$$

where $d_2(\bar{y},\beta)$ does not depend on n.

Note further that

$$(\alpha+n\bar{y})\tau_2(n) - n > (\alpha+n\bar{y})\left(\theta_n^*+n^{-\frac{1}{3}-\delta}\right) - n$$

$$= (\alpha+n\bar{y})\left(\frac{1}{\bar{y}} + O\left(n^{-\frac{1}{3}}\right)\right) - n$$

$$= O(n^{2/3}) \quad . \tag{A.64}$$

Therefore, combining (A.62), (A.63), and (A.64), we can write

$$M_n(\theta_n^*) + \log(\int_0^{\tau_1(n)} g(\theta)d\theta) \leq n \log\left(\frac{\tau_1(n)}{\theta^*(n)}\right) - (\theta_n^*-\tau_1(n))(\alpha+n\bar{y})$$

$$+ \log d_1(\bar{y},\beta)$$

and

$$M_n(\theta_n^*) + \log\left(\int_{\tau_2(n)}^{\infty} g(\theta)d\theta)\right) \leq n \log\left(\frac{\tau_2(n)}{\theta^*(n)}\right) - (\theta_n^*-\tau_2(n))(\alpha+n\bar{y})$$

$$+ \log d_2(\bar{y},\beta) + \frac{2}{3}\log n + \log c \quad , \tag{A.65}$$

where c is a suitable constant. Accordingly, we consider the expression, for $i = 1,2$,

$$n \log\left(\frac{\tau_i(n)}{\theta^*(n)}\right) - (\theta_n^*-\tau_i(n))(\alpha+n\bar{y})$$

$$= n \log\left(1 + \frac{\tau_i(n)-\theta^*(n)}{\theta_n^*}\right) - (\theta_n^*-\tau_i(n))(\alpha+n\bar{y})$$

$$= n\left(\frac{\tau_i(n)-\theta^*(n)}{\theta_n^*} - \frac{(\tau_i(n)-\theta^*(n))^2}{2\theta_n^*}\right) - (\theta_n^*-\tau_i(n))(\alpha+n\bar{y})$$

$$+ O((\tau_i-\theta^*(n))^3) \quad .$$

Now, since $\theta_n^* = \frac{1}{\bar{y}} + O(n^{-1})$, we have

$$n \log\left(\frac{\tau_i(n)}{\theta^*(n)}\right) - (\theta_n^*-\tau_i(n))(\alpha+n\bar{y})$$

$$= n((\tau_i(n)-\theta^*(n))(\bar{y}+0(n^{-1})$$

$$- n\left(\frac{\tau_i(n)-\theta^*(n))^2}{2}\left(\frac{1}{\bar{y}} + 0(n^{-1})\right)\right.$$

$$\left.- \alpha(\theta_n^*-\tau_i(n) - n\bar{y}(\theta_n^*-\tau_i(n))\right)$$

$$+ 0((\tau_i-\theta^*n)^3)$$

$$= -n\,\frac{(\tau_i(n)-\theta^*(n))^2}{2\bar{y}} + 0\left(n^{-\frac{1}{3}-\delta}\right) \quad . \tag{A.66}$$

Thus

$$\int_0^{\tau_1(n)} g(\theta)d\theta/e^{-M_n(\theta_n^*)} \leq e^{-\frac{n^{\frac{1}{3}-2\delta}}{4\bar{y}}} \tag{A.67}$$

and

$$\int_{\tau_2(n)}^{\infty} g(\theta)d\theta/e^{-M_n(\theta_n^*)} \leq e^{-\frac{n^{\frac{1}{3}-3\delta}}{4\bar{y}}} \tag{A.68}$$

for n sufficiently large.

Combining (A.52), (A.54), (A.60), (A.67) and (A.68), we have

$$e^{M_n(\theta_n^*)}\left|\int_0^{\infty} g(\theta)d\theta - \int_{-\infty}^{\infty} h_k(\theta)d\theta\right| = 0\left(n^{-\frac{k+1}{6}-\frac{1}{2}+\delta}\right) \quad .$$

Since

$$\int_{-\infty}^{\infty} h_k(\theta)d\theta = e^{-M_n(\theta_n^*)}\int_{-\infty}^{\infty} e^{-\frac{\gamma_n}{2}(\theta-\theta_n^*)^2}\left(1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}(\theta-\theta_n^*)^{\ell}\right)d\theta$$

$$= \frac{e^{-M_n(\theta_n^*)}}{\sqrt{\gamma_n}}\int_{-\infty}^{\infty} e^{-\frac{(\theta-\theta_n^*)^2}{2}}\left(1 + \sum_{\ell=3}^{k} K_{nk}^{(\ell)}\,\frac{(\theta-\theta_n^*)^{\ell}}{\gamma_n^{\ell/2}}\right)d\theta$$

$$= \sqrt{\frac{2\pi}{\gamma_n}}\, e^{-M_n(\theta_n^*)}\sum_{r=0}^{k} K_{nk}^{(\ell)}\mu_r/\gamma_n^{r/2} \quad , \tag{A.69}$$

thus establishing the theorem.

APPENDIX III

From the results of Appendix II, the various estimates of section 2 all have asymptotic estimates given by ratios of the form,

$$\hat{R}_{\alpha,\beta,y} = \int_{-\infty}^{\infty} h_k(\theta,\alpha+1,\beta)d\theta \Big/ \int_{-\infty}^{\infty} h_k(\theta;\alpha,\beta)d\theta \quad . \tag{A.70}$$

We now proceed to obtain an asymptotic expansion for this ratio.

Since $h_k(\theta)$ has the form

$$e^{-M_n(\theta_n^*) - \frac{\gamma_n}{2}(\theta-\theta_n^*)^2} \sum_{r=0}^{k} K_r(\theta_n^*)(\theta-\theta_n^*)^r$$

we have

$$\int_{-\infty}^{\infty} h_k(\theta)d\theta$$

$$= e^{-M_n(\theta_n^*)} \sqrt{\frac{2\pi}{\gamma_n}} \sum_{r=0}^{k} K_{nk}^{(r)}(\theta_n^*)\mu_r/\gamma_n^{r/2} \quad ,$$

where μ_r are the central moments of the normal distribution with variance unity. Now $M_n(\theta_n^*)$, γ_n, and θ_n^* depend on α. Thus, from (A.14) and (A.70)

$$\hat{R}_{\alpha,\beta,y} \sim e^{-M_n(\theta_n^*(\alpha+1,\beta)) + M_n(\theta_n^*(\alpha,\beta))} \sqrt{\frac{\gamma_n(\alpha,\beta)}{\gamma_n(\alpha+1,\beta)}} \cdot$$

$$\frac{\sum_{r=0}^{k} K_{nk}^{(r)}(\theta_n^*(\alpha+1,\beta)\mu_r/(\gamma_n(\alpha+1,\beta)^{r/2}}{\sum_{r=0}^{k} K_{nk}^{(r)}(\theta_n^*(\alpha,\beta)\mu_r/\gamma_n(\alpha,\beta))^{r/2}} \left(1 + 0\left(n^{-\frac{k-7}{6}+\delta}\right)\right) \quad . \tag{A.71}$$

We now employ the results of Appendix II to evaluate (A.71). From (A.15)

$$M_n(\theta_n^*(\alpha,\beta)) - M_n(\theta_n^*(\alpha+1,\beta)) =$$

$$(\theta_n^*(\alpha,\beta) - \theta_n^*(\alpha+1,\beta))(\alpha+n\bar{y}) - \theta_n^*(\alpha+1,\beta)$$

$$+ n \log \frac{\theta_n^*(\alpha+1,\beta)}{\theta_n^*(\alpha,\beta)} + (\beta-1) \log\left(\frac{1-e^{-\theta_n^*(\alpha+1,\beta)}}{1-e^{-\theta_n^*(\alpha,\beta)}}\right) . \tag{A.72}$$

Using (A.30), direct calculations establish

$$\theta_n^*(\alpha,\beta) - \theta_n^*(\alpha+1,\beta) =$$

$$\frac{1}{n\bar{y}^2} + \frac{1}{n^2\bar{y}^3}\left(\frac{2(\beta-1)}{1-e^{-1/\bar{y}}} + 1 - 2(\alpha+\beta)\right)$$

$$- \frac{(\beta-1)e^{-1/\bar{y}}}{n^2\bar{y}^4\left(1-e^{-1/\bar{y}}\right)^2} + O(n^{-3}) . \tag{A.73}$$

Similarly,

$$\frac{\theta_n^*(\alpha,\beta)}{\theta_n^*(\alpha+1,\beta)} = 1 + \frac{1}{n\bar{y}} + \frac{1}{n^2\bar{y}^2}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right)$$

$$- \frac{(\beta-1)e^{-1/y}}{n^2\bar{y}^3\left(1-e^{-1/\bar{y}}\right)^2} + O(n^{-3}) . \tag{A.74}$$

Thus,

$$n \log(\theta_n^*(\alpha,\beta)/\theta_n^*(\alpha+1,\beta)) =$$

$$\frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right) - \frac{(\beta-1)e^{-1/\bar{y}}}{n\bar{y}^3\left(1-e^{-1/\bar{y}}\right)^2} - \frac{1}{2n\bar{y}^2}$$

$$+ O(n^{-2}) . \tag{A.75}$$

Now write

$$1-e^{-\theta_n^*(\alpha,\beta)} = (1 - e^{-1/\bar{y}}) + \frac{e^{-1/\bar{y}}}{n\bar{y}^2}\left(\frac{(\beta-1)}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right) + 0(n^{-2}) \tag{A.76}$$

and replacing α by $\alpha+1$ in (A.76) produces $1-e^{-\theta_n^*(\alpha+1,\beta)}$.

Rewrite (A.76) as

$$(1-e^{-1/\bar{y}})\left(1 + \frac{e^{-1/\bar{y}}}{n(1-e^{-1/\bar{y}})\bar{y}^2}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right) + 0(n^{-2})\right) .$$

Then

$$\log\left(1-e^{-\theta_n^*(\alpha,\beta)}\right) = \log(1-e^{-1/\bar{y}}) + \frac{e^{-1/\bar{y}}}{n(1-e^{-1/\bar{y}})\bar{y}^2}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right) + 0(n^{-2}) .$$

Thus,

$$\log\left\{\frac{\left(1-e^{-\theta_n^*(\alpha+1,\beta)}\right)}{\left(1-e^{-\theta_n^*(\alpha,\beta)}\right)}\right\} = \frac{-e^{-1/\bar{y}}}{n\bar{y}^2\left(1-e^{-1/\bar{y}}\right)} + 0(n^{-2}) . \tag{A.77}$$

Consequently,

$$M_n(\theta_n^*(\alpha,\beta) - M_n(\theta_n^*(\alpha+1,\beta)) =$$

$$-\frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}\left(\alpha + \frac{1}{2}\right) - \frac{(\beta-1)e^{-1/\bar{y}}}{n\bar{y}^2\left(1-e^{-1/\bar{y}}\right)} + 0(n^{-2}) . \tag{A.78}$$

We now evaluate $\gamma_n(\alpha,\beta)/\gamma_n(\alpha+1,\beta)$. Since

$$n\gamma_n(\alpha,\beta) = \frac{1}{(\theta_n^*(\alpha,\beta))^2} - \frac{\beta-1}{n}\left[\left(1-e^{-\theta_n^*(\alpha,\beta)}\right)^{-2} - \left(1-e^{-\theta_n^*(\alpha,\beta)}\right)^{-1}\right] ,$$

from (A.30)

$$(\theta_n^*(\alpha,\beta))^{-2} = \bar{y}^2\ 1 + \frac{2}{n\bar{y}}\left(\frac{\beta-1}{1-e^{-1/\bar{y}}} - (\alpha+\beta-1)\right)^{-1} + 0(n^{-2}) ,$$

and

$$\frac{\beta-1}{n}\left(\left(1-e^{-\theta_n^*(\alpha,\beta)}\right)^{-2} - \left(1-e^{-\theta_n^*(\alpha,\beta)}\right)^{-1}\right)$$

$$= \frac{\beta-1}{n}\left[\left(1-e^{-1/\bar{y}}\right)^{-2} - \left(1-e^{-1/\bar{y}}\right)^{-1}\right] + 0(n^{-2}) \quad ,$$

we set

$$\gamma_n(\alpha,\beta)/\gamma_n(\alpha+1,\beta) = 1 - \frac{2}{n\bar{y}} + 0(n^{-2})$$

and

$$\left(\gamma_n(\alpha,\beta)/\gamma_n(\alpha+1,\beta)\right)^{\frac{1}{2}} = 1 - \frac{1}{n\bar{y}} + 0(n^{-2}) \quad . \tag{A.79}$$

Finally, we set $k = 5$, obtaining

$$\sum_{r=0}^{k} K_{nk}^{(r)}(\theta_n^*(\alpha,\beta))\mu_r/\left(\gamma_n(\alpha,\beta)\right)^{r/2}$$

$$= 1 - M_n^{(4)}(\theta_n^*(\alpha,\beta))/8\gamma_n^2(\alpha,\beta) \quad . \tag{A.80}$$

From (A.19), (A.30) and (A.79), we see that

$$\frac{1-M_n^{(4)}\left(\theta_n^*(\alpha+1,\beta)\right)/8\gamma_n^2(\alpha+1,\beta)}{1-M_n^{(4)}\left(\theta_n^*(\alpha,\beta)\right)/8\gamma_n^2(\alpha,\beta)} = 1 + 0(n^{-2}) \quad . \tag{A.81}$$

Combining (A.71), (A.76), (A.77), (A.78), (A.79) and (A.80), we have

$$\hat{R}_{\alpha,\beta,y} = e^{-\frac{1}{\bar{y}} + \frac{1}{n\bar{y}^2}(\alpha+\frac{1}{2}) - \frac{(\beta-1)^{-1/\bar{y}}}{n\bar{y}^2(1-e^{-1/\bar{y}})}}\left(1 - \frac{1}{n\bar{y}}\right)\left(1 + 0(n^{-2})\right) \quad . \tag{A.82}$$

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In decision theoretic approaches to estimation problems, loss functions of the type $L(\theta,\hat{\theta}) = |\theta-\hat{\theta}|^{\alpha}$, $\alpha > 0$ are often employed; often $\alpha = 2$. In many applications of reliability and life testing, such loss functions are inappropriate. Alternative loss functions which appear to be more suited to the intended application are proposed and Bayesian estimates of the exponential parameter are obtained for these.

Asymptotic expansions of such estimators are given and compared with estimators given in the literature.

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